SIMPLE, REDUCIBLE VENN DIAGRAMS ON FIVE CURVES AND HAMILTONIAN CYCLES

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Abstract In [3,4], using graph theory, we developed procedures for the construction of Venn diagrams. Utilizing these procedures with some new methods that we introduce here, we determine the number of simple, reducible spherical Venn diagrams on five sets. In so doing, we obtain examples of Venn diagrams which yield answers to several problems and conjectures of Grünbaum. Among others we construct a simple, reducible Venn diagram with five congruent ellipses. We show that this diagram is unique on the sphere and produces two different plane diagrams. This corrects some erroneous statements that started with John Venn more than a century ago in 1880 and have been repeated frequently by others since then.

Key words Planar and spherical Venn diagram, convex and exposed Venn diagram, Venn graph or dual graph, planar graph.
MSC (1991) number(s): 05C10, 05C45 and 51M04.

1. Introduction This paper is inspired by Grünbaum's conjectures and problems on simple Venn diagrams [6]. Our objective is not the enumeration of simple, reducible spherical Venn diagrams with 5 curves, but rather a way to study the geometrical and topological properties of simple Venn diagrams with five curves such as convexity, spherical and planar embedding, and drawability with special curves: triangles, rectangles and congruent ellipses.

Recently Venn diagrams have been used in computerized design, automated industrial manufacturing, and geometric and solid modeling. In computerized geometric and solid modeling, algorithms are known only for convex diagrams. Therefore the characterizations of those diagrams are no longer merely nice curiosities of geometry.

In Section 2, the exact definitions used in this paper are given. Here, we also state the facts from [3,4], and [6] which are needed for our results. In Section 3, we list those questions of Grünbaum which we are able to answer or partially answer. In Section 4, we enumerate simple, reducible Venn diagrams with five curves on the sphere, and we enumerate convex, simple, reducible nonisomorphic Venn diagrams with five curves in the plane. By doing this we will answer Problem 3.1 in the simple, reducible case. We also answer negatively Conjectures 3.2 and 3.3. We prove that though every simple, reducible Venn diagram with five curves is in the same class as some exposed diagram, there are nevertheless simple, reducible Venn diagrams that are not in the same class as any convex one.

Finally, we prove that there are exactly two simple, reducible Venn diagrams with five congruent ellipses, and show that they are in the same class. The first four examples of irreducible Venn diagrams with five congruent ellipses were obtained by Grünbaum [7], a nonsimple one by Schwenk [10], and an another irreducible one by the authors and K. B. Chilakamarri [3]. Those examples together with the two diagrams constructed here, finally correct the error which due to Venn and perpetuated by Grünbaum in the simple, reducible case. Grünbaum says [6]:

> "In [11, 12], Venn gave examples of Venn diagrams with four ellipses. However, he mistakenly stated that no five ellipses can form a Venn diagram; indeed, it takes only a little patience to verify that the five congruent ellipses in each part of [the figure] form a Venn diagram. Venn's erroneous assertion was repeated -- unchecked and unchallenged -- by several authors[1] for almost a century. The first Venn diagram of five ellipses ... was published only in 1975 [7]; a non-simple example ... was found by Schwenk [10]. Using Euler's theorem and the fact that two ellipses can intersect in no more than four points, it follows easily (by an argument similar to the one concerning circles) that there can be no Venn diagrams with six or more ellipses. One possible explanation for Venn's error is that he may have believed that all Venn diagrams can be constructed following a sort of 'greedy algorithm' as described above: to get a diagram with n curves first make a diagram with n-1 curves and then add the last one. However, it is easy to verify that **none of the Venn**

[1]Among others, in the article **Logic Diagrams** in P. Edwards (editor), 1967, **Encyclopedia of Philosophy**, Macmillan, New York, 1967, Logic diagrams by M. Gardner, Vol. 5 pp. 77-81.

> **diagrams in [the figure] (nor any other simple Venn diagram of five ellipses) can be obtained by adding a fifth ellipse to a Venn diagram of four ellipses** [emphasis added]. Probably the same is true without the assumption of simplicity."

For a more detailed history of this problem see [9].

2. Definitions and Notation To make this paper self contained we list the definitions we need, and the theorems that were proved in [3] and [6] which are used in this paper. We use graph theory throughout; for terminology the reader is referred to [1] or any other standard book on graph theory.

The infinite face of a planar graph is called the **outer face.** A face with n edges will be called an **n-face**, and n is the **size** of the face. A planar graph is **maximal bipartite** if and only if all its faces are 4-faces [3].

We follow Grünbaum [6] in the terminology of Venn diagrams. A **Venn diagram** (or n-Venn diagram) in the plane is a collection of simple closed Jordan curves $\mathscr{F} = \{C_1, C_2, \ldots, C_n\}$ such that each of the 2^n sets $X_1 \cap X_2 \cap \ldots \cap X_n$ is a nonempty and connected region; here, X_i is either the bounded interior or the unbounded exterior of C_i, $i = 1,2,\ldots,n$.

A Venn diagram is a **simple Venn diagram** if at most two curves intersect (transversally) at any point in the plane.

The projection of a Venn diagram from the plane to the sphere via stereographic projection yields a **spherical Venn diagram.** Two Venn diagrams in the plane which can be projected to the same spherical Venn diagram will be said to belong to the same **class.** Two Venn diagrams are **isomorphic** (or of the **same type**) if, by continuous transformation of the plane, one of them can be changed into the other or its mirror image.

A Venn diagram is said to be **exposed** if each of its curves has an arc on the boundary of the unbounded region. A Venn diagram is called a **convex Venn diagram** if it is isomorphic to a Venn diagram formed by convex curves.

A Venn diagram $\mathscr{F}$ with n curves is called **irreducible** if each of the n families of n-1 curves, obtained from $\mathscr{F}$ by deleting in turn one of the n curves, fails to be a Venn diagram. Otherwise it is called **reducible.**

We can associate two distinct graphs with any Venn diagram on a set of n curves $\mathscr{F} = \{C_1, C_2, \ldots, C_n\}$. The Venn diagram itself can be viewed as a plane graph $V(\mathscr{F})$ where all the intersection points of the curves in $\mathscr{F}$ are the vertices and the edges are the segments of the curves with vertices as the end points. The second graph we wish to associate with $\mathscr{F}$ is the dual graph of $V(\mathscr{F})$, denoted by $D(\mathscr{F})$. We will call the graphs $V(\mathscr{F})$ and $D(\mathscr{F})$ the **Venn diagram** (or **n-Venn diagram** to indicate the number of curves) and **Venn graph,** respectively. Note that the definition of isomorphism of Venn diagrams given above is not the same as the usual graph isomorphism.

The Venn graph $D(\mathcal{F})$ is simple, while the Venn diagram $V(\mathcal{F})$ may have multiple edges. The graphs $V(\mathcal{F})$ and $D(\mathcal{F})$ are both planar graphs. An important observation is that if $\mathcal{F}$ forms a simple Venn diagram, then each face of $D(\mathcal{F})$ is a quadrilateral, and hence $D(\mathcal{F})$ is a maximal bipartite planar graph.

In [3] we proved that: a simple Venn diagram with $|\mathcal{F}| \geq 3$ is 3-connected, and a simple Venn graph $D(\mathcal{F})$ is 3-connected if $|\mathcal{F}| \geq 3$.

Grünbaum stated the following [6]: "It is not hard to prove that if a Venn diagram is convex then it is exposed. However, the conjecture that every exposed and simple diagram is convex is disproved by [the figure]. (In this example, the interiors of two of the curves intersect in a set with two connected components, hence no isomorphism to a diagram with convex curves is possible.)"

If in a Venn diagram the interiors of two curves intersect in a set with two (or more) connected components we will say that the Venn diagram has a **Grünbaum configuration.** An example is shown in Figure 1. A **forbidden configuration** in a Venn diagram is two curves such that all the four possible intersections of the interiors and exteriors form a Grünbaum configuration. It is easy to see that if a Venn diagram has a forbidden configuration then any graph-isomorphic copy of the Venn diagram has a Grünbaum configuration, and thus there is no graph-isomorphic copy having a convex drawing in the plane. An example of a forbidden configuration is shown in Figure 2.

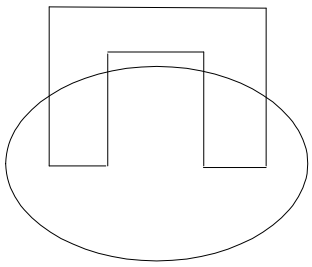


Figure 1

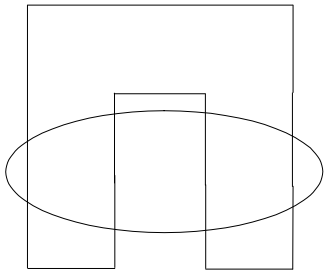


Figure 2

3. Conjectures and Problems Posed by Grünbaum Grünbaum [6] raised several problems and conjectures dealing with the number of Venn diagrams and the existence of certain kinds of Venn diagrams. We will mention those problems and conjectures which we are able to answer or partially answer.

Problem 3.1 It is not known how many types of simple Venn diagrams exist with five (or more) curves.

Conjecture 3.2 Every simple and convex Venn diagram with five curves is isomorphic to one of the diagrams in Figure 7 of [6].

Conjecture 3.3 Every convex Venn diagram with five curves is isomorphic to one with five congruent ellipses.

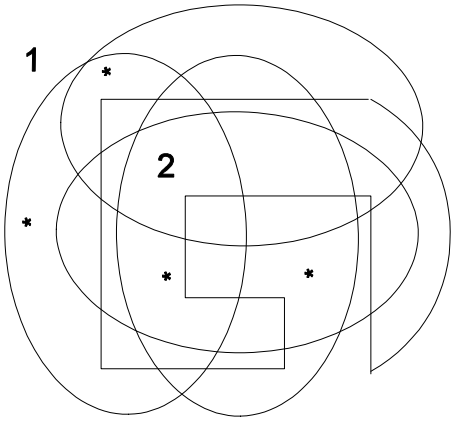
Problem 3.4 Is every simple Venn diagram of the same class as some exposed diagram, or even some convex diagram?

We showed in [3] that, for $n < 5$, every simple Venn diagram is in the same class as some simple convex (and consequently, exposed) diagram; and for $n > 5$, there exist simple Venn diagrams that are not in the same class as any simple exposed or convex diagram. Consequently, the case $n = 5$ deserves special consideration. In [3] Figure 20 shows a simple 5-Venn diagram that is exposed, but is not in the same class as any convex diagram.

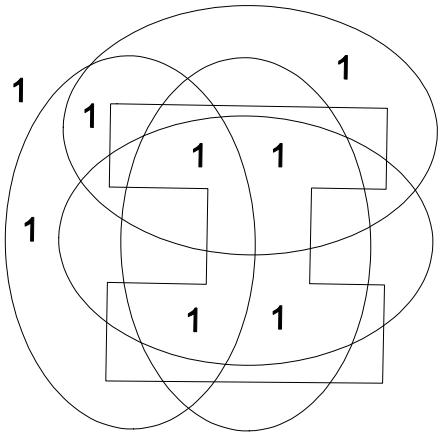
4. Enumeration of Venn Diagrams on 5 curves In this section we will enumerate simple, reducible spherical Venn diagrams with five curves and we also will enumerate nonisomorphic convex, simple, reducible, Venn diagrams in the plane.

Theorem 4.1 (i) There are eleven distinct simple, reducible spherical 5-Venn diagrams shown in Figures 3 and 4.
(ii) The diagrams of (i) each have a representation in the plane as an exposed diagram, but five of them (shown in Figure 4) cannot

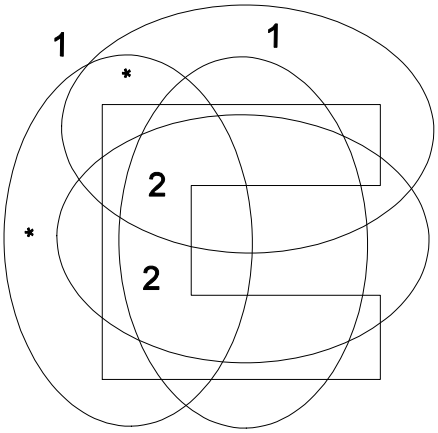
be represented as a convex one.
(iii) The eleven spherical diagrams yield twelve nonisomorphic convex, simple, reducible, 5-Venn diagrams in the plane. Each of them but two can be drawn with five triangles, the remaining two can be drawn with rectangles (shown in Figure 5.a and 5.b).
(iv) Out of the twelve convex diagrams there are exactly two nonisomorphic convex, simple, reducible Venn diagrams that can be drawn in the plane with five congruent ellipses . They are in the same class (shown in Figure 6).



The clown



Diamond



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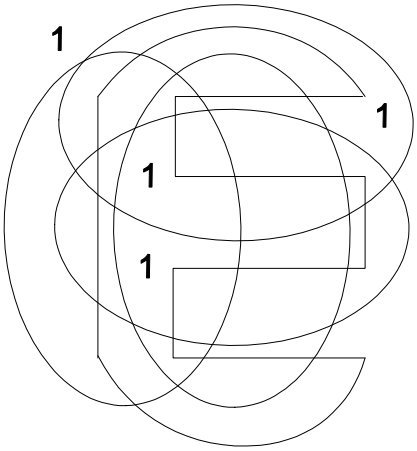

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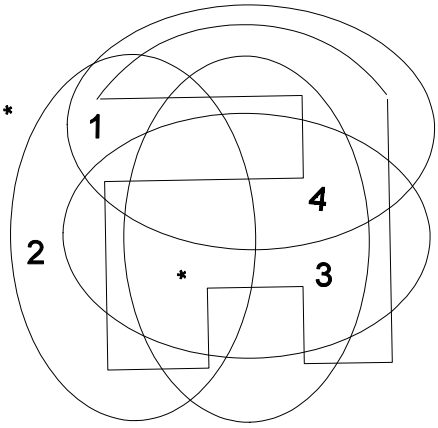


Chair

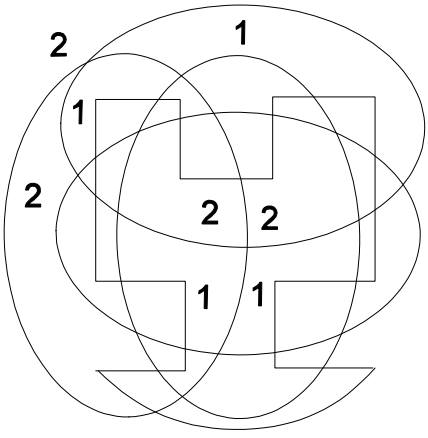


Figure 3

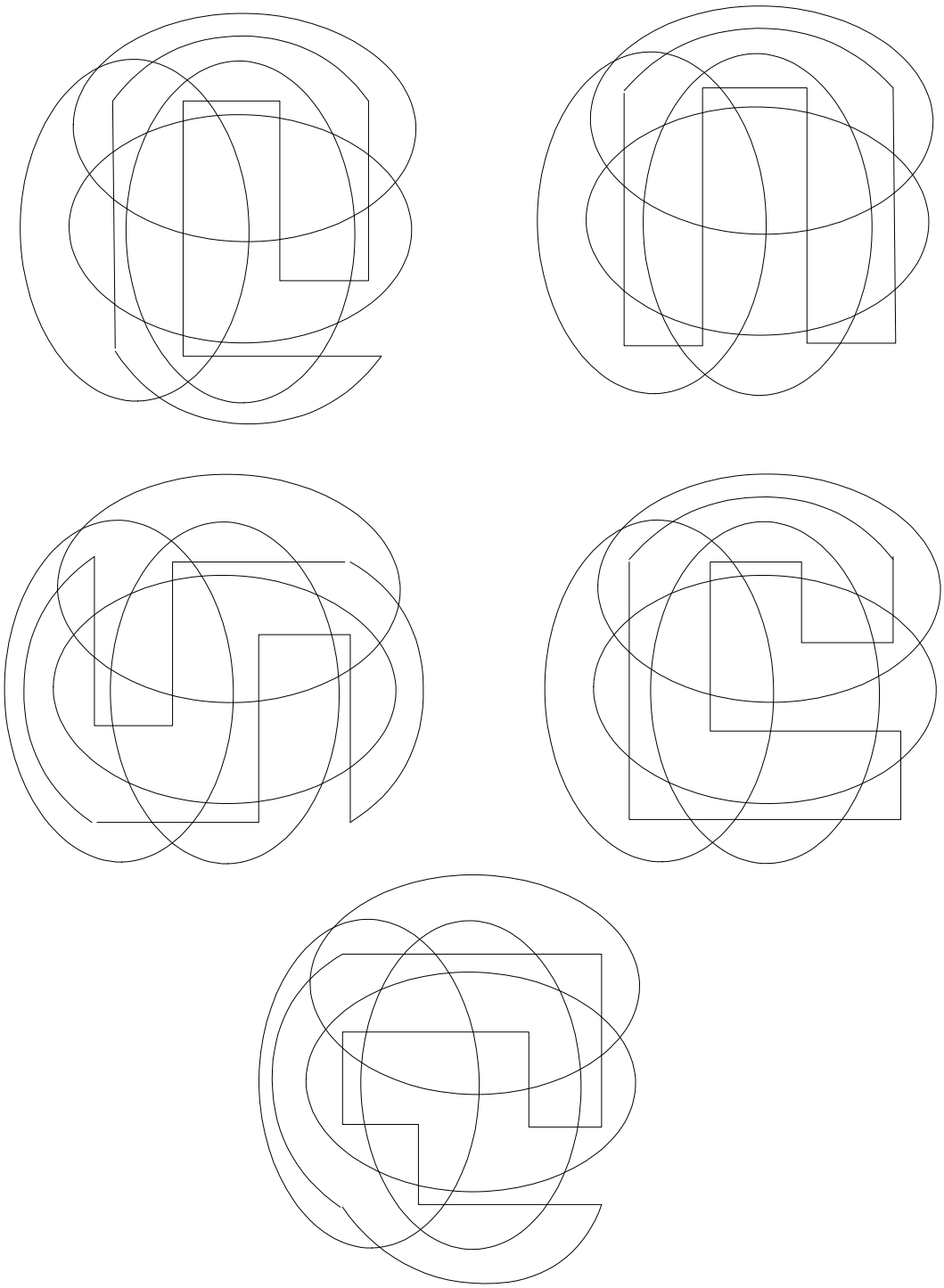
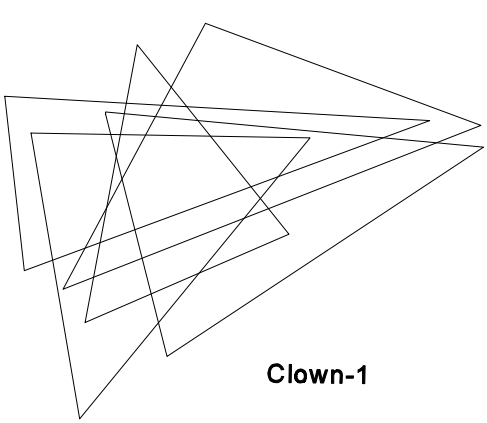


Figure 4



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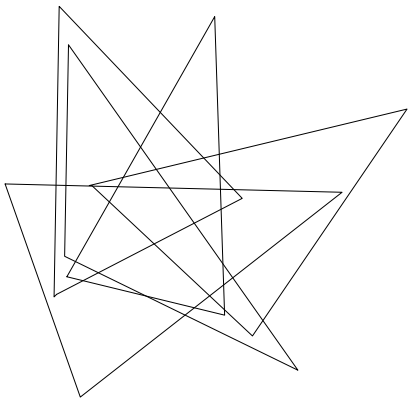

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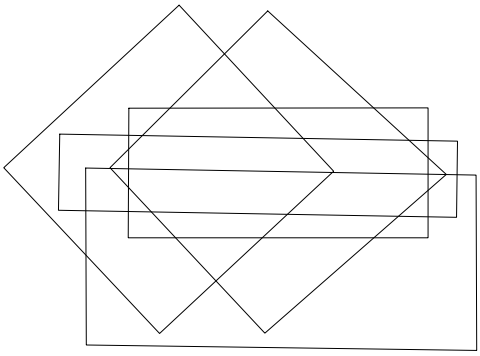
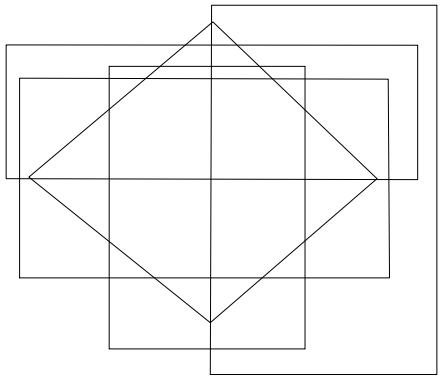
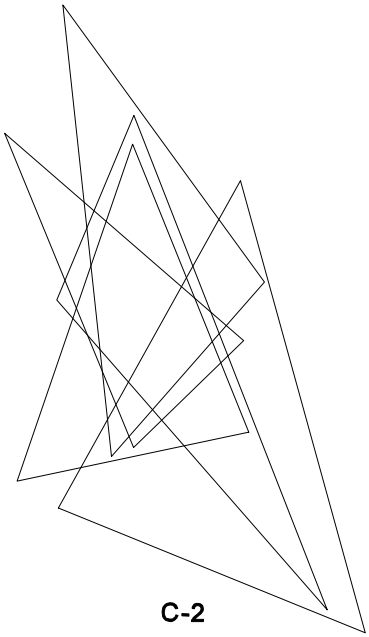


Diamond



C-1



C-2

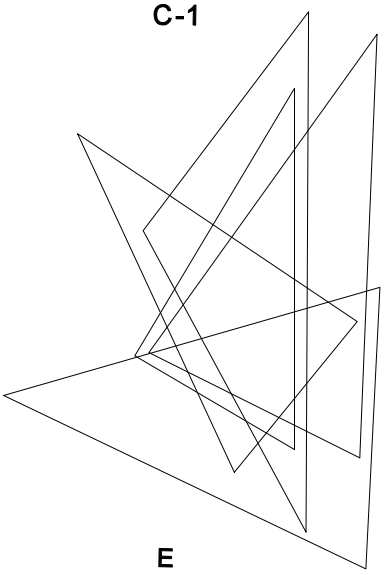


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Figure 5.a

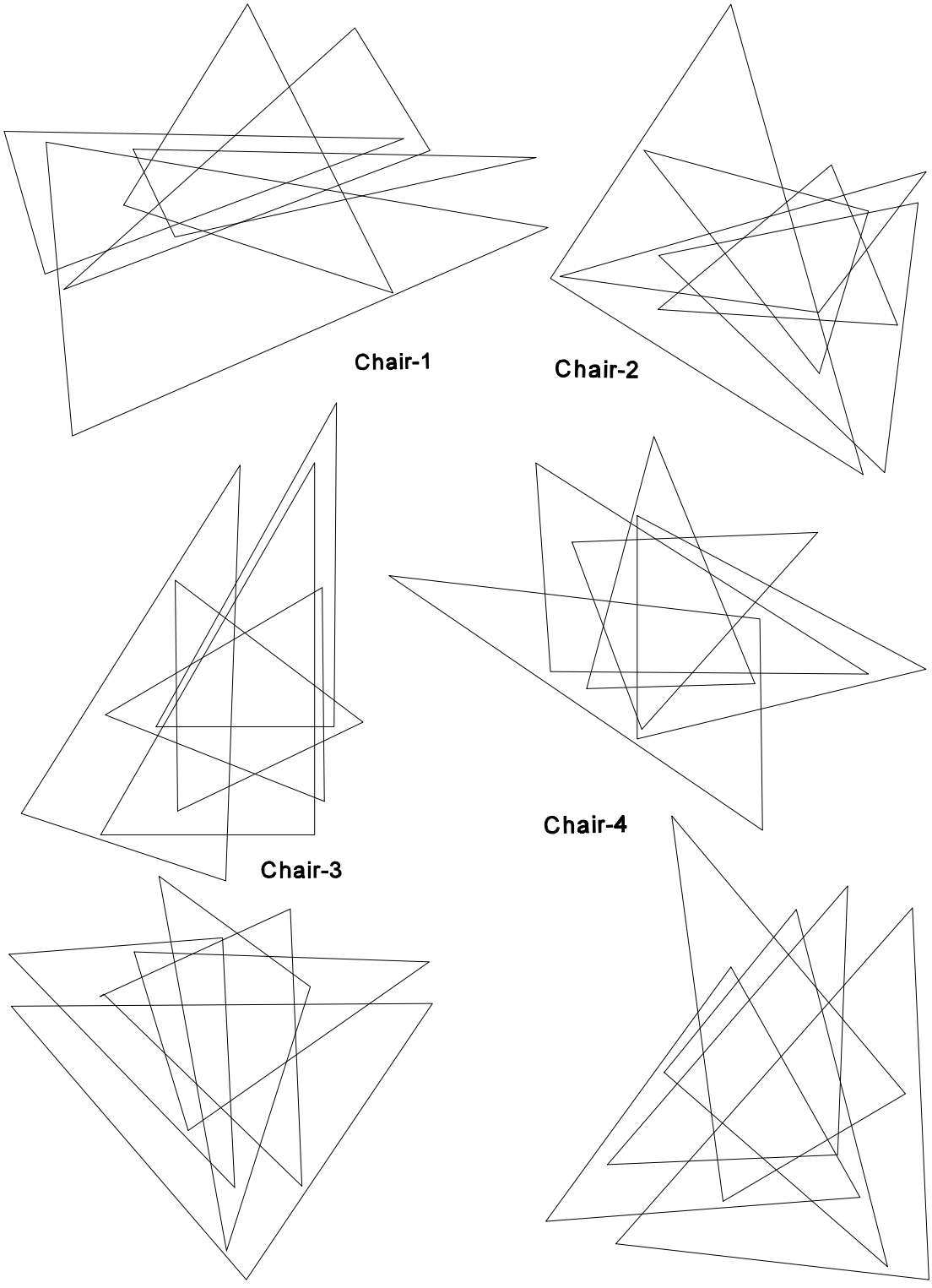

Figure 5.b

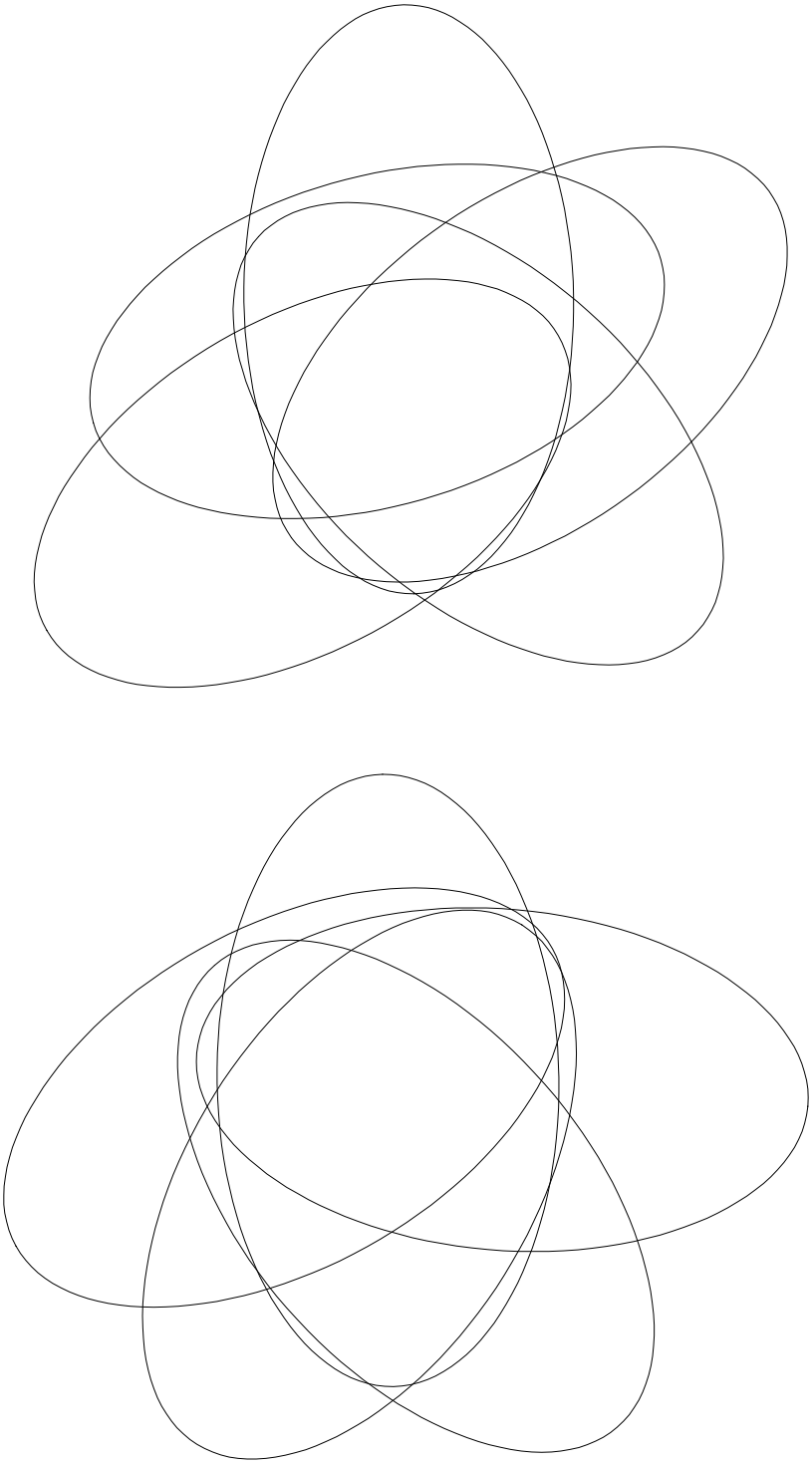

Figure 6

Proof We break the proof of Theorem 4.1 into a series of observations and lemmas. First we prove (i).

Since every reducible n–Venn diagram can be obtained from an (n–1)–Venn diagram with the addition of a suitable curve, to prove (i) it is enough to find all curves that extend all 4–Venn diagrams. We proved in [3] that there exists only one spherical 4–Venn diagram, so we need only find all curves that extend this diagram. Winkler [13] proved that for n > 1, a simple n–Venn

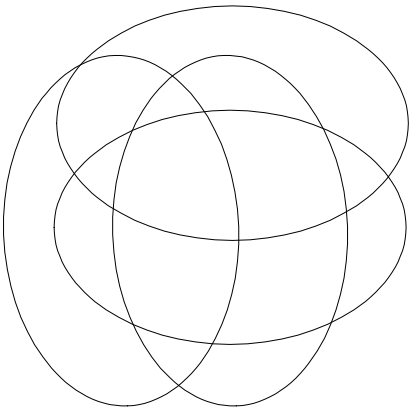

Figure 7

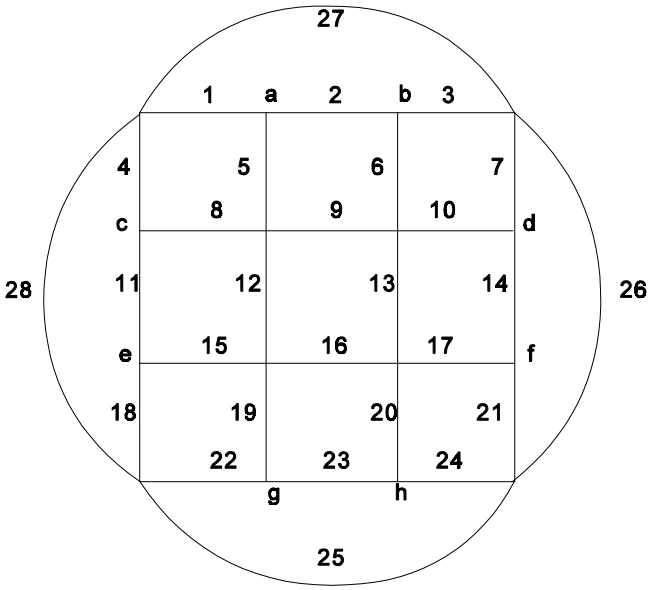

Figure 8

diagram is extendible to a simple (n+1)–Venn diagram if and only if its Venn graph is Hamiltonian. (There is a generalization of this theorem in [2]; see also [4].) Therefore to prove (i) it is enough to find all Hamiltonian cycles of the spherical Venn graph (Figure 7) of the unique spherical 4–Venn diagram shown in Figure 8. The following lemma does this.

Lemma 4.2 The spherical Venn graph of the unique simple spherical 4–Venn diagram has eleven distinct Hamiltonian cycles (up to symmetry on the sphere).

Proof We first determine all Hamiltonian cycles of a planar drawing of the Venn graph with a specified outer face, up to rotational symmetry in the plane. Since the Venn graph is 3–connected [3], Whitney's Theorem [14] guarantees a unique embedding in the plane once the outer face is specified. We then identify those cycles which are equivalent under spherical symmetry, using the fact that the two faces of the Venn graph containing only vertices of degree 4 are equivalent on the sphere. We label the edges of the Venn graph with numbers and all of the vertices of degree 3 with letters as shown in Figure 8.

We first observe that at least two of the edges 2, 11, 14, and 23 must be included in any Hamiltonian cycle, for no adjacent pair can be missing: Say 2 and 14 are missing; then the edges 6, 3, 7, and 10 must be included, forming a 4–cycle.

There are thus three main cases to consider (by symmetry): the Hamiltonian cycle, which we shall denote by C, contains two opposite edges, three edges, or all four edges.

Case 1 C contains two opposite edges. By symmetry we assume the edges 2 and 23 are in the Hamilton cycle, and 11 and 14 are not.

Immediately, this forces edges 7, 10, 17, 21, 15, 18, 4, and 8 to be in C. Then 9 and 16 must be excluded because they would force a small cycle. We consider edges 12 and 13.

Subcase 1a Edges 12 and 13 are in C. C is

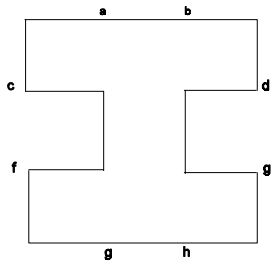

Figure 9

Subcase 1b Edge 12 is in C (by symmetry). C is

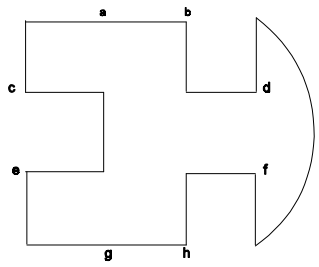

Figure 10

Subcase 1c Neither 12 nor 13 is in C. C is

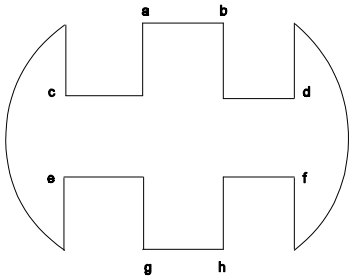

Figure 11

Case 2 C contains three edges. By symmetry, we may assume 2, 11, and 23 are in C and 14 is not. Then 7, 10, 17, and 21 must also be in C. We consider the middle edges 9, 12, 13, and 16. Clearly no more than three can be in C.

Subcase 2a Three are in C. They must be 9, 12, and 16 for otherwise there would be a vertex of degree three in C. C is

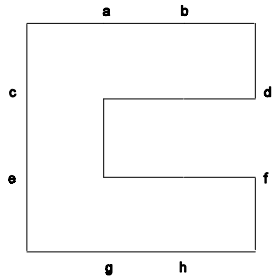

Figure 12

Subcase 2b Two are in C.

Subcase 2b(i) The opposite pair 9 and 16 are in C. C is

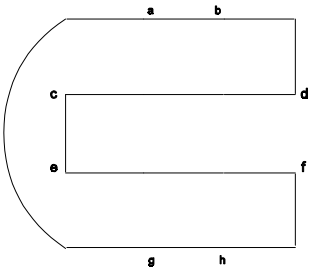

Figure 13

Subcase 2b(ii) The opposite pair 12 and 13 are in C. Then exactly one of 5 and 19 must be in C, say 5 by symmetry. C is

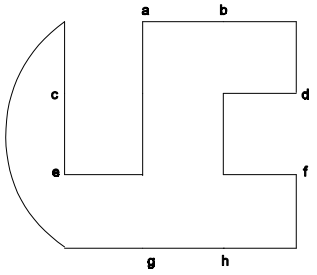

Figure 14

Subcase 2b(iii) An adjacent pair is in C. Degree two requirements exclude 13, so we may assume by symmetry that they are 9 and 12. C is

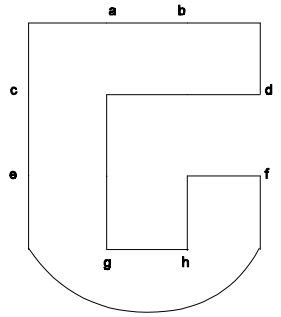

Figure 15

Subcase 2c One is in C. Edge 13 forces a small cycle.

Subcase 2c(i) Edge 9 (or by symmetry 16) is in C. C is

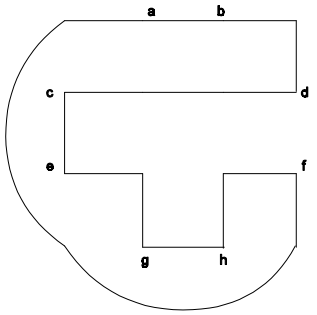

Figure 16

Subcase 2c(ii) Edge 12 is in C. At least one of 5 or 19 must be in C or we get a 4-cycle. If one, say 5, is in C, then C is

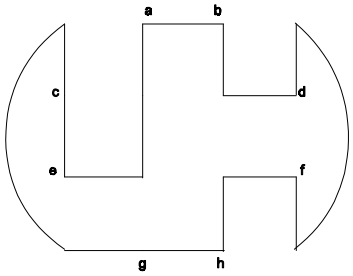

Figure 17

If both are in C, then C is

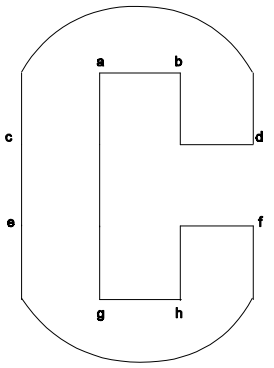

Figure 18

Subcase 2d None is in C. C is

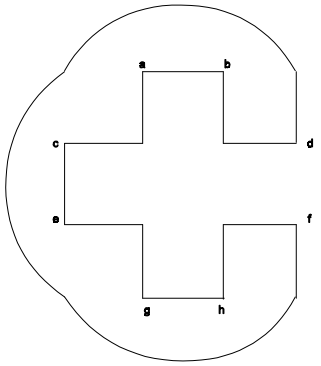

Figure 19

Case 3 All four are in C. There must be at least one of the "middle" edges 9, 12, 13, 16 in C, for otherwise a small cycle is forced. Clearly, there can be at most three.

Subcase 3a Three are in C, say 9, 12, and 13 by symmetry. Exactly one of 15 or 17, say 15 by symmetry, must be in C or a small cycle is forced. C is

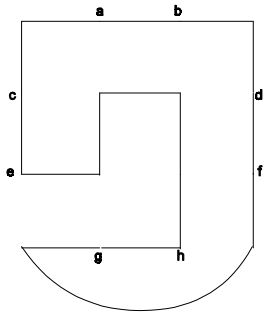

Figure 20

Subcase 3b Two are in C.

Subcase 3b(i) An opposite pair, say 9 and 16 by symmetry, is in C. To avoid small cycles, at most one of 5 and 6 and one of 19 and 20 can be in C, but at least one of the four must be in C.

Subcase 3b(i)I Edges 5 and 19 (by symmetry) are in C. C is

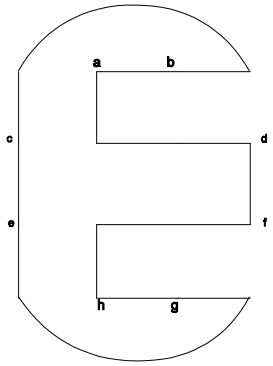

Figure 21

Subcase 3b(i)II Edge 5 and 20 (by symmetry) are in C. C is

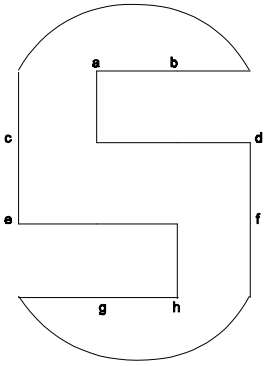

Figure 22

Subcase 3b(i)III Only 5 is in C (by symmetry). C is

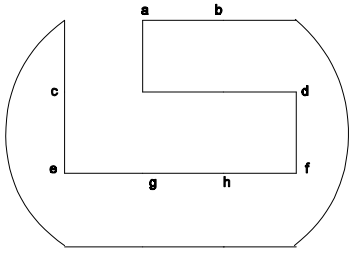

Figure 23

Subcase 3b(ii) An adjacent pair, say 9 and 12 by symmetry, is in C.

Subcase 3b(ii)I Edge 3 is in C. C is

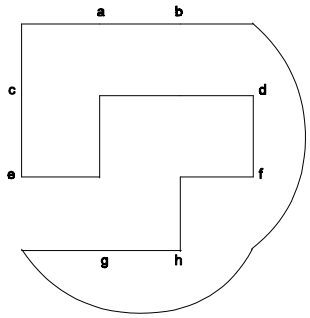

Figure 24

Subcase 3b(ii)II Edge 3 is not in C but edge 6 is in C. C is

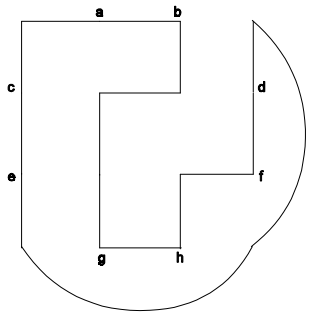

Figure 25

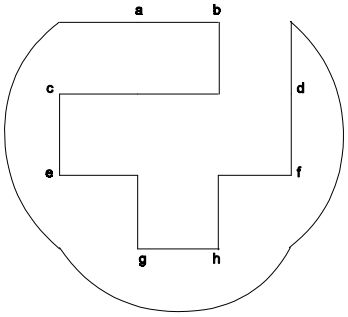

Figure 26

Subcase 3c One is in C, say 9 by symmetry. This forces exactly one of 8 or 10 to be in C, for otherwise we obtain a small cycle. By symmetry, we assume edge 8 is in C. Then C is

We now determine which of these eighteen cycles are equivalent on the sphere. Redrawing the Venn graph in Figure 8 with the center face as the outer face shows that the cycles in the pairs of Figures 9 and 11, 12 and 19, 13 and 18, 14 and 17, 15 and 16, 20 and 26, 24 and 25 are equivalent. We shall call them **twins**. The cycles of Figure 10, 21, 22, and 23 are their own twins and we call them **identical-twins**. (This process is quite easy and is left to the reader.) The result is the eleven Hamiltonian cycles, and thus eleven simple reducible 5-Venn diagrams on the sphere, as shown in Figures 3 and 4. ∎

That the eleven diagrams in Figures 3 and 4 are all different can be checked easily by counting the 5-faces, together with the number of intersections of each of the curves with all of the others. This completes the proof of the Theorem 4.1 (i).

Since each diagram in Figures 3 and 4 is drawn as an exposed diagram we need only to prove the second statement of (ii). But it is easy to find in each of the diagrams of Figure 4 an ellipse that forms a forbidden configuration with the added Hamiltonian cycle, thus they cannot be in the same class as any convex diagram.

To prove (iii), we first need to recall the fact from Section 2 that a convex drawing of a Venn diagram in the plane must be exposed [6]. It follows immediately that distinct spherical Venn diagrams cannot have isomorphic drawings in the plane. In Figure 3 we marked the 5-faces of each diagram with an asterisk or with a number. The 5-faces are marked with the same number if the drawings of the diagram in the plane using the 5-faces as the outer face yield isomorphic Venn diagrams. If we draw a diagram with a 5-face marked with an asterisk as an outer face in the plane then at least one of the ellipses with the Hamiltonian cycle form a Grünbaum

configuration in this drawing, and thus it cannot be convex. (This can be checked easily in Figure 3.) To show that the 5-faces that are marked with the same numbers in a diagram yield isomorphic drawings in the plane, and those that are marked with different numbers in a diagram yield nonisomorphic drawings, we proceed as follows. To each diagram we associate a 5-tuple in which the first number represents the number of intersections of the Hamiltonian cycle with the four ellipses, and the other four numbers represent the number of intersections of the remaining ellipses with the other four curves in the following order; the second and the third numbers are associated with the left vertical and with the right vertical ellipses, while the fourth and fifth numbers are associated with the upper horizontal and the lower horizontal ellipses respectively.

In the diagram "The clown", there are two 5-faces marked with numbers 1 and 2, and four others with an asterisk. The associated 5-tuple is (16, 10, 12, 10, 12). Since only the Hamiltonian cycle has 16 intersections, any isomorphic drawing must map the cycle to itself. The Hamiltonian cycle in face 1 is adjacent with two curves having intersections 10 and 12, while in face 2 it is adjacent with two curves having intersections 10 and 10, therefore they are nonisomorphic.

The diagram "Diamond" has associated 5-tuple (16, 8, 16, 8, 12), and eight 5-faces. It is obvious from Figure 3 (by symmetry) that the four 5-faces in the center marked with 1 are isomorphic. If we draw the diagram with the twin Hamiltonian cycle (Figure 11), then the other four 5-faces are in the center, and again by symmetry they are isomorphic. (Verification is left to the reader.) To see that this diagram has only a unique convex drawing in the plane we need to mention that the mapping that interchanges the two curves with 16 and 8 intersections respectively, is an isomorphic mapping of the two drawings with the different twins.

The diagram "C" has two 5-faces marked with an asterisk, two marked with 1, and two marked with 2. The associated 5-tuple is (16, 10, 14, 8, 12). From Figure 3 by symmetry again, it is easy to see that the 5-faces marked with the same numbers yield isomorphic drawings. That faces 1 and 2 give nonisomorphic drawings follows from the fact that there is only one curve with 16 intersections, and this is adjacent in face 1 with two curves having intersections 12 and 14, but in face 2 with two curves having intersections 8 and 10.

By symmetry in Figure 3, and from the fact that diagram "E" has an identical-twin, it follows immediately that it has only one convex drawing in the plane.

The diagram "Chair" has two 5-faces marked with an asterisk, and four marked with four different numbers. The 5-tuple is (16, 10, 12, 8, 14). That the diagram provides at least 3 nonisomorphic drawings follows immediately again from the fact that there is only one curve with 16 intersections, and this curve in the different 5-faces is adjacent with two curves having intersections 10 and 12, 14 and 12, 10 and 14, and finally 10 and 14 again. To see that the last two drawings are nonisomorphic

requires only checking the order of the intersections of the Hamiltonian cycle and we leave this once more for the reader.

Finally, that the diagram "Goal post" has two distinct drawings is an immediate consequence of the facts that this diagram has an identical-twins cycle and a 5-tuple (16, 8, 14, 8, 14), the symmetry of the drawings in Figure 3, and the fact that the unique curve with 16 intersections in face 1 is adjacent with curves having intersections 8 and 14, while in face 2 they are 14 and 14.

To complete the proof of Theorem 4.1 (iii) we note that all but two of the distinct convex Venn diagrams are drawn with five triangles in Figure 5.a and 5.b. The remaining two Venn diagrams are drawn with five rectangles in Figure 5.a. To see that they cannot be drawn with five triangles we observe that there is an ellipse which intersects the added Hamiltonian cycle eight times in the diagram "Diamond", but two triangles can have at most six intersections. Furthermore, in the drawing of diagram "C" in Figure 3, there are three curves, the right vertical ellipse, the lower horizontal ellipse, and the Hamiltonian cycle, that intersect each others as shown in Figure 27.

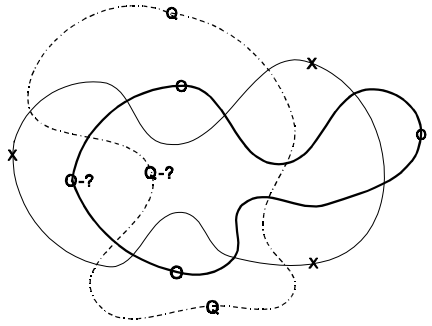

Figure 27

If these three curves are to be drawn with three convex polygons, then the three polygons must have a vertex at each marked segment. (The vertices in different curves are marked with O, Q, and X respectively.) Indeed, if any two convex (or nonconvex) polygons meet each other in two consecutive intersections then there must be a vertex on one of the curves between the intersections. In one of the segments marked by O-? and Q-? a vertex must be placed. But this vertex either would be the fourth one in the curve marked with O's, or would provide a nonconvex angle at the vertex marked with Q-?. This shows that the diagram cannot be drawn with five triangles.

To prove Theorem 4.1 (iv), we first observe that in Figure 3 each diagram but one (The clown) has an ellipse that meets the Hamiltonian cycle more than four times and thus cannot be drawn

with ellipses. In Figure 6 we draw the two distinct Venn diagrams (The clown-1 and clown-2 in Figure 5.a) with five congruent ellipses, using Drawperfect 1.1. The following equations of five congruent ellipses form an example of the first Venn diagram in Figure 6. (It is not difficult to find an example for the other one.) The first ellipse has the equation $x^2 + y^2/4 = 1$. The other four equations can be derived from the first equation by transformations. The first number in each 4-tuple denotes the transformation in the x direction, the second in the y direction, and the last the rotation in degrees; (-26, 16, 95), (-70, -145, 127), (28, -83, 32), and (60, -31, -43). The horizontal and vertical transformations are scaled by 0.009. This completes the proof of Theorem 4.1. ∎

5. Answers to Grünbaum's problems and conjectures Theorem 4.1 answers Grünbaum's Problem 3.1 on the sphere in the case of reducible, simple Venn diagrams, and also in the plane for convex, simple, reducible Venn diagrams. In a forthcoming paper [5] we plan to answer this problem fully on the sphere, and for convex Venn diagrams on the plane. We do not address the problem in the case of planar, nonconvex diagrams due to the fact that there are too many such diagrams, and furthermore such investigation will not provide us with additional information on their geometrical and topological properties. Figures 5.a and 5.b show twelve distinct counterexamples for Grünbaum's Conjecture 3.2. Another one was published in [3], and in the paper [5] we will show that there are no more. All of the diagrams except the first two give a counterexample to Grünbaum's Conjecture 3.3. These diagrams are closely related to some other conjectures and problems of Grünbaum [8] that we have not discussed here. That two of the diagrams in Figure 5.a cannot be drawn with triangles was a surprise. The diagram "C-1" was a bigger surprise than the diagram "Diamond", since the obstacle in "C-1" is not the number of intersections of two curves, but the topological relation of three curves. In Figure 4 we show five simple, reducible 5-Venn diagrams that are not in the same class as any convex diagram. In [5] we also will show that, in contrast to the reducible case, every convex, irreducible, simple Venn diagram is isomorphic to one with five congruent ellipses. Finally, our greatest surprise was in Figure 6: two simple, reducible 5-Venn diagrams with five congruent ellipses, which were supposed not to exist.

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